EXERCISESET 7, TOPOLOGY IN PHYSICS

- The hand-in exercise is exercise 2.
- Please hand it in electronically at `topologyinphysics2019@gmail.com` (1 pdf!)
- Deadline is Wednesday May 1, 23.59.
- Please make sure your name and the week number are present in the file name.

Exercise 1: Supersymmetry. Let $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ be a $\mathbb{Z}_2$-graded Hilbert space.

a) Show that the supertrace vanishes on supercommutators:

$$\mathrm{Tr}_s([A, B]) = 0.$$

b) Let $H = Q^2$ be the Hamiltonian of a supersymmetric quantum mechanical system, and denote by $\mathcal{H}_n$ its n-th eigenspace with eigenvalue λ_n, where $\mathcal{H}_0$ corresponds to the zero modes, i.e., $\lambda_0 = 0$. Show that $\mathcal{H}_n = \mathcal{H}_n^+ \oplus \mathcal{H}_n^-$ and that for $n > 0$, $\mathcal{H}_n^+ \cong \mathcal{H}_n^-$.

c) Use $b)$ to show that the *Witten index*

$$\mathrm{Tr}_s(e^{-\beta H}),$$

equals the number $\dim \mathcal{H}_0^+ - \dim \mathcal{H}_0^-$. Also, show the β-independence by taking the derivative w.r.t. β.

$\star$ **Exercise 2: The symbol of a differential operator.** Let D be a differential operator of degree k on a manifold M acting on functions (i.e., not sections of a vector bundle). In local coordinates $x = (x^1, \ldots, x^n)$ we have

$$D = \sum_{|I| \leq k} c_I(x) \frac{\partial}{\partial x^I}$$

a) Show that the *symbol* of D,

$$\sigma(D)(x, \xi) := \sum_{|I|=k} c_I(x) \xi_I,$$

is coordinate invariant if we interpret ξ_i, $i = 1, \ldots, n$ as coordinates on the cotangent bundle T^*M induced by x^i: $\alpha = \sum_i \xi_i dx^i$ for $\alpha \in T^*M$. In other words: the symbol $\sigma(D)$ is a smooth function on T^*M!

b) Show that the *full symbol*

$$\sigma_f(D)(x, \xi) := \sum_{|I| \leq k} c_I(x) \frac{\partial}{\partial x^I}$$

is coordinate dependent. This is the reason to only consider the top order part, which is sometimes called simply *symbol*, but also often *principal symbol*.

c) Show that the symbol can alternatively be found by the formula

$$\sigma(D)(x,\xi) = \frac{1}{k!}\left(D(f^k)\right)(x)$$

where $f \in C^\infty(M)$ such that $f(x) = 0$ and $d_x f = \xi$. Note that this expression is inherently coordinate independent.

d) Show that for a differential operator $D : \Gamma(E) \to \Gamma(F)$ acting on sections of vector bundles not much changes and the symbol is a section of the vector bundle $\pi^*\mathrm{Hom}(E,F)$ over T^*M.

e) Compute the symbol of a Dirac operator and show that over a *Riemannian* manifold a Dirac operator is always elliptic.

Exercise 3: The heat kernel of the harmonic oscillator. Consider the Hamiltonian of the harmonic oscillator

$$H_x = -\frac{d^2}{dx^2} + a^2x^2, \quad a > 0,$$

acting on $L^2(\mathbb{R})$. The associated heat operator e^{-tH} can be represented by a smooth function $k_t(x,y),\ t > 0,\ x, y \in \mathbb{R}$ satisfying

- $k_t(x,y)$ is symmetric in x and y,
- $k_t(x,y)$ satisfies the heat equation

$$\frac{\partial k}{\partial t} + H_x k = 0,$$

- initial conditions are given by

$$\lim_{t\downarrow 0} k_t(x,y) = \delta(x-y).$$

Using the most general Gaussian function

$$k_t(x,y) = \exp(\alpha_t x^2 + \beta_t xy + \alpha_t y^2 + \gamma_t),$$

as ansatz, write down a system of ODE's for the coefficient functions α, β, γ. Solve these equations to show that

$$k_t(x,y) = (4\pi t)^{-1/2}\left(\frac{2at}{\sinh 2at}\right)^{1/2} \exp\left(-\frac{1}{4t}\left[\frac{2at}{\tanh 2at}(x^2+y^2) - \frac{2at}{\sinh 2at}(2xy)\right]\right).$$